ME 360 Control Systems
Examples – Partial Fraction Expansions

1. Given: $X(s) = \dfrac{5s+3}{(s+1)(s+2)(s+3)}$

   Find: a) the partial fraction expansion of $X(s)$, and b) $x(t)$.

   Solution: Given the characteristic equation has real, unequal roots, the partial fraction expansion has the form

   $$X(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

   where the coefficients $K_i \ (i = 1,2,3)$

   $$K_1 = \left[(s+1)X(s)\right]_{s=-1} = \left(\frac{5s+3}{(s+2)(s+3)}\right)_{s=-1} = -1$$

   $$K_2 = \left[(s+2)X(s)\right]_{s=-2} = \left(\frac{5s+3}{(s+1)(s+3)}\right)_{s=-2} = 7$$

   $$K_3 = \left[(s+3)X(s)\right]_{s=-3} = \left(\frac{5s+3}{(s+1)(s+2)}\right)_{s=-3} = -6$$

   So, the partial fraction expansion is

   $$\boxed{X(s) = \left(\frac{-1}{s+1}\right) + \left(\frac{7}{s+2}\right) + \left(\frac{-6}{s+3}\right)}$$

   Using the Laplace transform table gives

   $$\boxed{x(t) = -e^{-t} + 7e^{-2t} - 6e^{-3t}}$$

2. Given: $X(s) = \dfrac{5s+15}{(s+2)(s^2+3s+9)}$

   Find: a) the partial fraction expansion of $X(s)$, and b) $x(t)$.

   Solution: Given the characteristic equation has one real root and a pair of complex conjugate roots, the partial fraction expansion is of the form

   $$X(s) = \left(\frac{K}{s+2}\right) + \left(\frac{As+B}{s^2+3s+9}\right)$$

where the coefficient K may be found as before

$$K = \left[(s+2)X(s)\right]_{s=-2} = \left[\frac{5s+15}{s^2+3s+9}\right]_{s=-2} = \frac{5}{7}$$

and the coefficients A and B are found by clearing fractions as follows

$$\begin{aligned} 5s+15 &= K(s^2+3s+9)+(As+B)(s+2) \\ &= (K+A)s^2+(2A+B+3K)s+(9K+2B) \end{aligned}$$

Equating the coefficients of the powers of s on both sides of the equation gives

$$\begin{aligned} A+K &= 0 && (s^2) \\ 2A+B+3K &= 5 && (s^1) \\ 9K+2B &= 15 && (s^0) \end{aligned}$$

Using the first and third equations, we find $A = -\frac{5}{7}$ and $B = \frac{30}{7}$. So, the partial fraction expansion is

$$X(s) = \tfrac{5}{7}\left(\frac{1}{s+2}\right) - \tfrac{5}{7}\left(\frac{s-6}{s^2+3s+9}\right) = \tfrac{5}{7}\left(\frac{1}{s+2}\right) - \tfrac{5}{7}\left(\frac{s+(-6)}{(s+\frac{3}{2})^2+\frac{27}{4}}\right)$$

Using #4 and #18 in the Laplace transform tables gives

$$x(t) = \tfrac{5}{7}\left[e^{-2t} - 3.055e^{-1.5t}\sin(2.5981t+\phi)\right] \qquad \phi = \tan^{-1}\left(\frac{2.5981}{-6-1.5}\right) = \begin{cases} -0.3335 \ \text{(rad)} \\ \text{or} \\ 2.8081 \ \text{(rad)} \end{cases}$$

The correct choice of ϕ must be made to satisfy the initial conditions of the differential equation (not given here).

3. <u>Given</u>: $X(s) = \dfrac{s^5+a_1s^4+a_2s^3+a_3s^2+a_4s+a_5}{s^6+11s^5+79s^4+427s^3+1510s^2+3800s+4000}$

<u>Find</u>: What is the form of $x(t)$? Identify the steady-state and transient parts of $x(t)$.

<u>Solution</u>: Using the root solving feature on your calculator, the poles of $X(s)$ are found to be $\pm 5j$, $-2\pm 3.4641j$, -5, and -2. So, the partial fraction expansion and $x(t)$ have the forms

$$X(s) = \left(\frac{K_1}{s+2}\right)+\left(\frac{K_2}{s+5}\right)+\left(\frac{A_1s+B_1}{s^2+25}\right)+\left(\frac{A_2s+B_2}{s^2+4s+16}\right)$$

$$x(t) = \underbrace{K_1e^{-2t}+K_2e^{-5t}+C_1e^{-2t}\sin\left(\sqrt{12}\,t+\phi_1\right)}_{\text{transient response}} + \underbrace{C_2\sin(5t+\phi_2)}_{\text{steady-state response}}$$