ME 360 Control Systems
Transfer Functions

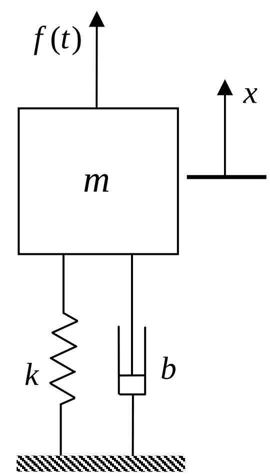
<u>Single-Input, Single-Output (SISO) Systems</u>

- For ***linear*** systems that have a single input and a single output, we can define a single ***transfer function*** that quantifies the dynamic behavior of the system.

- Mathematically, the transfer function is defined as the ***Laplace transform of the output divided by the Laplace transform of the input***, assuming all initial values are zero.

- As an example, consider the single degree-of-freedom mass, spring, damper system shown with a forcing function $f(t)$. For this system, we know that ***the differential equation of motion*** is

$$\boxed{m\ddot{x} + b\dot{x} + kx = f(t)}$$



- ***Applying Laplace transforms*** to both sides of the equation and ***assuming that all initial values are zero***, we get

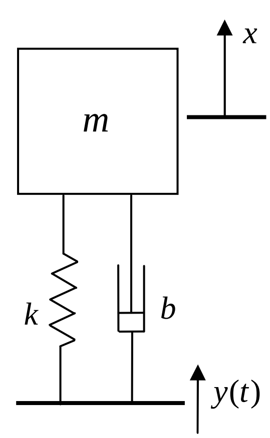
$$(ms^2 + bs + k)X(s) = F(s)$$

Given that the ***input*** is $f(t)$ and the ***output*** is $x(t)$, the system transfer function is defined to be

$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}}$$

- If the base motion $y(t)$ is the system input, then the differential equation of motion is

$$\boxed{m\ddot{x} + b\dot{x} + kx = b\dot{y} + ky}$$



- Applying Laplace transforms to this equation gives

$$(ms^2 + bs + k)X(s) = (bs + k)Y(s)$$

Given the ***input*** is $y(t)$ and the ***output*** is $x(t)$, the ***system transfer function*** is

$$\boxed{\frac{X(s)}{Y(s)} = \frac{bs + k}{ms^2 + bs + k}}$$

- If the input to the system is the ***impulse function***, $\delta(t)$, then $X(s)$ the Laplace transform of the response is equal the transfer function, since $F(s) = \boldsymbol{L}(\delta(t)) = 1$. So, the transfer function describes the ***impulse response*** of the system.

- In general, a transfer function will have the form

$$\frac{X(s)}{F(s)} = \frac{s^m + b_m s^{m-1} + b_{m-1} s^{m-2} + \cdots + b_1}{s^n + a_n s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_1}$$

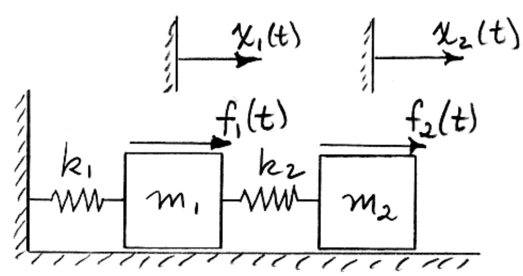
- The roots of the numerator are called the ***zeros*** of the system, and the roots of the denominator are called the ***poles*** of the system. In this class, we will assume that the order of the numerator is less than the order of the denominator, that is, $m < n$.

<u>Multiple-Input, Multiple-Output (MIMO) Systems</u>

- For ***linear*** systems with ***M input variables*** and ***N output variables***, we define $M \times N$ transfer functions, one relating each input/output pair. Together, these transfer functions can be used to quantify the behavior of the system. As before, the transfer function is defined as the Laplace transform of the output variable divided by the Laplace transform of the input variable, assuming all initial values are zero.

- As an ***example***, consider the two degree-of-freedom mass, spring system shown with forcing functions $f_1(t)$ and $f_2(t)$. It can be shown that the equations of motion for this system may be written as

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = f_1(t)$$
$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = f_2(t)$$



- Applying Laplace transforms to these equations and writing the resulting equations in ***matrix form*** gives

$$\begin{bmatrix} m_1 s^2 + k_1 + k_2 & -k_2 \\ -k_2 & m_2 s^2 + k_2 \end{bmatrix} \begin{Bmatrix} X_1(s) \\ X_2(s) \end{Bmatrix} = [A]\{X(s)\} = \begin{Bmatrix} F_1(s) \\ F_2(s) \end{Bmatrix} = \{F(s)\}$$

- From this equation, we define ***four transfer functions***. Note that for the transfer functions involving $F_1(s)$, we assume $F_2(s) = 0$, and for the transfer functions involving $F_2(s)$, we assume $F_1(s) = 0$.

$$\frac{X_1(s)}{F_1(s)}, \frac{X_1(s)}{F_2(s)}, \frac{X_2(s)}{F_1(s)}, \text{ and } \frac{X_2(s)}{F_2(s)}.$$

- ***Using Cramer's Rule we can solve the boxed equation*** above for $X_1(s)$ and $X_2(s)$:

$$X_1(s) = \frac{\det\begin{pmatrix} F_1(s) & -k_2 \\ F_2(s) & m_2 s^2 + k_2 \end{pmatrix}}{\det[A]}$$

$$= \left(\frac{m_2 s^2 + k_2}{\det[A]}\right) F_1(s) + \left(\frac{k_2}{\det[A]}\right) F_2(s)$$

where $\boxed{\det[A] = (m_1 s^2 + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2}$.

$$X_2(s) = \frac{\det\begin{pmatrix} m_1 s^2 + k_1 + k_2 & F_1(s) \\ -k_2 & F_2(s) \end{pmatrix}}{\det[A]}$$

$$= \left(\frac{k_2}{\det[A]}\right) F_1(s) + \left(\frac{m_1 s^2 + k_1 + k_2}{\det[A]}\right) F_2(s)$$

- From these results we can now define the ***four transfer functions*** of the system.

$$\boxed{\frac{X_1(s)}{F_1(s)} = \left(\frac{m_2 s^2 + k_2}{\det[A]}\right)} \qquad \boxed{\frac{X_1(s)}{F_2(s)} = \left(\frac{k_2}{\det[A]}\right)}$$

$$\boxed{\frac{X_2(s)}{F_1(s)} = \left(\frac{k_2}{\det[A]}\right)} \qquad \boxed{\frac{X_2(s)}{F_2(s)} = \left(\frac{m_1 s^2 + k_1 + k_2}{\det[A]}\right)}$$

<u>Experimental Determination of Transfer Functions</u>

- To measure transfer functions ***experimentally***, we use actuators to excite the system, sensors to measure the system input and response (output), and a ***data acquisition system*** to record the signals.

- MATLAB's ***system identification toolbox*** uses these signals to estimate the transfer function that relates the two.

- A ***dynamic signal analyzer*** can also be used. In addition to recording the system input and output signals, it can calculate their ***Fast Fourier Transforms*** (FFT's) and display the ratio (output/input) in the form of a Bode diagram. The Bode diagram is one way of graphically displaying a transfer function. We will discuss Bode diagrams later in the semester.